Numerical approaches to self-force calculations along scatter orbits

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Contents

- Intro to self-force and scattering
- Self-force by mode-sum regularization
- Scalar field toy model
  - Time-domain code
  - Frequency-domain code
- (Not) extending to gravity
- New approaches
  - Re-projection methods
  - Hyperboloidal methods
  - “Nature adores a vacuum”
- Gravitational fluxes
- Outlook

Numerical results in following talk by O. Long.

Gravitational scattering

Complementary approaches include:

- **Numerical relativity:** low-separation, short durations, ≈ equal mass.
- **Post-Minkowskian expansion (expansion in G):** weak-field, arbitrary mass-ratio, analytical.
- **Gravitational self-force (expansion in mass-ratio ϵ):** strong- and weak-field, small mass-ratio, *primarily* numerical.

What can we do with self-force?

- **Determine/validate PM coefficients:** already put into practice for scalar field [Barack et al 2023]; GSF will give exact PM results in future! [Damour 2020],
- **Benchmark/resum** other approaches in the strong-field e.g. [Long, Whittall & Barack 2024]
- **Improve waveform models:** e.g. incorporate SF into EOB via $\chi \to H$ mapping [Damour 2016]

Self-force expansion

Metric of the physical spacetime is expanded about background as a series in $\epsilon := \mu/M \ll 1$,

$$g^{\text{phys}}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots$$

- 0SF: Background metric $g_{\alpha\beta}$. Smaller object moves along fixed background geodesic.
- 1SF: Perturbation $h^{(1)}_{\alpha\beta}$ sourced by point particle on fixed background geodesic. Leading order conservative and dissipative self-forces $\propto \epsilon$.
- 2SF: Perturbation $h^{(2)}_{\alpha\beta}$ sourced by particle on 1SF-perturbed trajectory. Gives rise to additional self-force terms $\propto \epsilon^2$.

Particle description derived, not assumed.

1SF equation of motion

- Metric perturbation may be split into regular and singular fields,
  [Detweiler & Whiting 2003]

$$h_{\alpha\beta} = {h^{R}}_{\alpha\beta} + {h^{S}}_{\alpha\beta},$$

defined in terms of certain acausal Green's functions.

- Only $h^{R}_{\alpha\beta}$ contributes to the self-force. For example, at 1SF order,

$$\frac{Du^{\alpha}}{d\tau} = q\nabla^{\alpha\beta\gamma} h^{R(1)}_{\beta\gamma}\Big|_{z(\tau)} + O(q^2),$$

where

$$\nabla^{\alpha\beta\gamma} h_{\gamma\beta} := -\frac{1}{2}\left(g^{\alpha\beta} + u^{\alpha}u^{\beta}\right) u^{\gamma}u^{\delta}\left(2\nabla_{\delta}h_{\beta\gamma} - \nabla_{\beta}h_{\gamma\delta}\right).$$

Scalar-field toy model

- Scalar charge q with mass μ scattered off a Schwarzschild black hole of mass M. Scalar field:

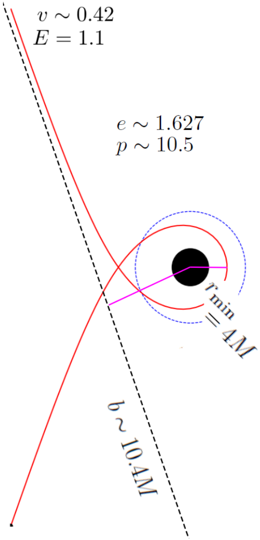
$$\nabla\Phi = -4\pi q \int \frac{\delta^4(x^\alpha - x_p^\alpha(\tau))}{\sqrt{-g}} d\tau,$$

  where $\epsilon := q^2/\mu M \ll 1$ is the expansion parameter.

- At leading order take $x_p^\alpha(\tau)$ to be a scatter geodesic: parameterised by e.g. velocity at infinity v and impact parameter b

- Particle feels a self-force due to interaction with its own scalar field:

$$u^\beta \nabla_\beta (\mu u^\alpha) = q \nabla_\beta \Phi^R := F^\alpha_{\mathrm{self}}.$$



Conservative and dissipative forces

- Self-force split into conservative and dissipative forces:

$$F^{\alpha}_{\rm cons} = \frac{1}{2}\left[F^{\alpha}_{\rm self}(\Phi^{\rm ret}) + F^{\alpha}_{\rm self}(\Phi^{\rm adv})\right],$$
$$F^{\alpha}_{\rm diss} = \frac{1}{2}\left[F^{\alpha}_{\rm self}(\Phi^{\rm ret}) - F^{\alpha}_{\rm self}(\Phi^{\rm adv})\right],$$

- Symmetries of Kerr geodesics relate advanced and retarded forces:
  [Mino 2003, Hinderer & Flanagan 2008]

$$F_{\alpha}^{\rm self(adv)}(\tau) = \epsilon_{\alpha} F_{\alpha}^{\rm self(ret)}(-\tau),$$

  where $\epsilon_{\alpha} = (-1, 1, 1, -1)$ and periapsis is at $\tau = 0$.

- Thus extract conservative/dissipative forces from retarded calculation alone.

Numerical self-force calculations: mode-sum regularisation

Decompose the field into spherical harmonics centred on the Schwarzschild black hole:

$$\Phi(t, r, \theta, \phi) = \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \psi_{\ell m}(t, r) Y_{\ell m}(\theta, \phi)$$

Self-force can be computed using mode-sum regularisation: [Barack & Ori 2000-03]

$$F_\alpha(\tau) = \sum_{\ell=0}^{\infty} \left\{ \nabla_\alpha \left[\frac{1}{r} \sum_{m=-\ell}^{+\ell} \psi_{\ell m} Y_{\ell m} \right]_{x_p^\pm(\tau)} \mp \left(\ell + \frac{1}{2} \right) \underbrace{A_\alpha(\tau) - B_\alpha(\tau)}_{\text{regularisation parameters}} \right\}$$

Time-domain

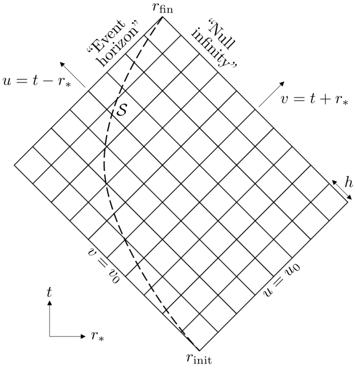
- Solve $(1+1)d$ PDEs for $\psi_{\ell m}(t, r)$.
- Scattering: [Long & Barack 2209.03740]

Frequency-domain

- Construct $\psi_{\ell m}(t, r)$ from frequency-modes, obtained by solving ODEs.
- Scattering: [Whittall & Barack 2305.09724]

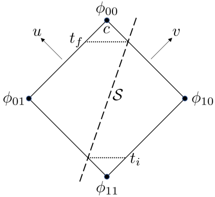
Field equation solved using finite differences on a characteristic grid

$$\frac{\partial^2 \psi_{\ell m}}{\partial u \partial v} + \frac{1}{4} V_\ell(r)\psi_{\ell m} = \bar{S}_{\ell m}\delta\left(r - r_p(t)\right)$$



- Distributional source implemented as a jump in non-vacuum cells:

$$\psi_{00} = Z - \psi_{11} + (\psi_{01} + \psi_{10})\left(1 - \frac{h^2}{8} V_\ell(r_c)\right) + O(h^3)$$



$$Z := \int_C \bar{S}_{\ell m}\delta\left(r - r_p(t)\right) dudv$$

- Characteristic initial conditions $\psi_{\ell m}(u_0, v) = 0 = \psi_{\ell m}(u, v_0) \rightarrow$ junk radiation.

[Figures from arXiv:2209.03740]

Frequency-domain decomposition

- Additionally decompose

$$\psi_{\ell m}(t,r) = \int_{-\infty}^{+\infty} \psi_{\ell m\omega}(r) e^{-i\omega t} d\omega.$$

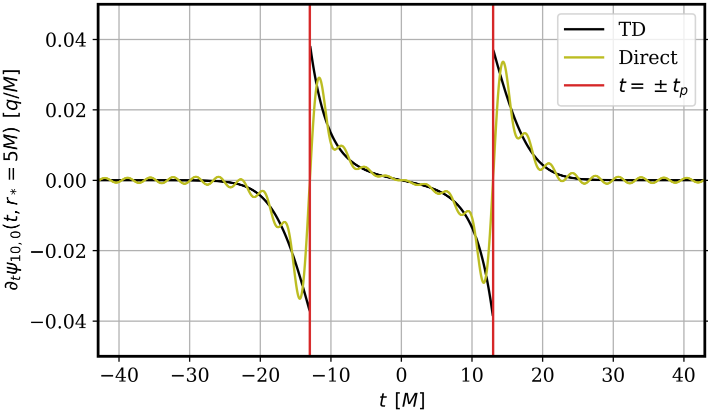
- Frequency-modes $\psi_{\ell m\omega}$ obey ODE

$$\frac{d^2\psi_{\ell m\omega}}{dr_*^2} - \left[V_\ell(r) - \omega^2\right]\psi_{\ell m\omega} = S_{\ell m\omega}(r)$$

- Retarded solution expressed in terms of homogeneous solution basis $\psi^{\pm}_{\ell\omega}(r)$ using variation of parameters:

$$\psi_{\ell m\omega}(r) = \psi^{+}_{\ell\omega}(r)\int_{r_{\min}}^{r} \frac{\psi^{-}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega} f(r')} dr' + \psi^{-}_{\ell\omega}(r)\int_{r}^{+\infty} \frac{\psi^{+}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega} f(r')} dr'$$

Direct reconstruction



Direct reconstruction $\psi_{\ell m}(t,r) \approx \int_{-\omega_{\max}}^{+\omega_{\max}} \psi_{\ell m\omega}(r) e^{-i\omega t} d\omega$ not practical due to Gibbs phenomenon.

EHS reconstruction [Barack, Ori & Sago 2008]

- EHS reconstruction: recover $\psi_{\ell m}(t, r)$ separately in $r \leq r_p(t)$ and $r \geq r_p(t)$ using homogeneous solutions.
- For example, field modes in the "internal" region $r \leq r_p(t)$ reconstructed from

$$\tilde{\psi}^{-}_{\ell m \omega}(r) := \psi^{-}_{\ell \omega}(r) \int_{r_{\min}}^{+\infty} \frac{\psi^{+}_{\ell \omega}(r') S_{\ell m \omega}(r')}{W_{\ell \omega} f(r')} dr'.$$

- Restores exponential, uniform convergence.
- External reconstruction: EHS cannot be applied in $r > r_p(t)$ for scatter orbits.
- Cancellation problem: significant loss of precision at high eccentricities [van de Meent 2016]. Limited to $r_p \sim r_{\min}$ for scatter orbits.

Frequency-domain code [Whittall & Barack 2305.09724]

- FD code makes use of internal EHS reconstruction and one-sided mode-sum regularisation.
- Calculation of "normalisation integral"

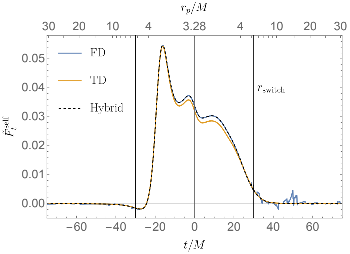
$$C^{-}_{\ell m\omega} := \int_{r_{\min}}^{+\infty} \frac{\psi^{+}_{\ell\omega}(r') S_{\ell m\omega}(r')}{W_{\ell\omega} f(r')} dr'.$$

  is numerically challenging:
  - Significant error from truncating at finite $r_{\max} \to$ IBP.
  - Integrand highly oscillatory, extremely slow! $\to$ specialist quadrature.
- Adaptive mode-sum truncation: routine to detect anomalous ℓ-mode behaviour caused by cancellation, set appropriate $\ell_{\max}$.
  - Accuracy deteriorates rapidly.

Comparing the codes

- FD code more precise than TD code near to periapsis along strong-field orbits.
- FD code can access $\ell > 15$ modes near to periapsis.

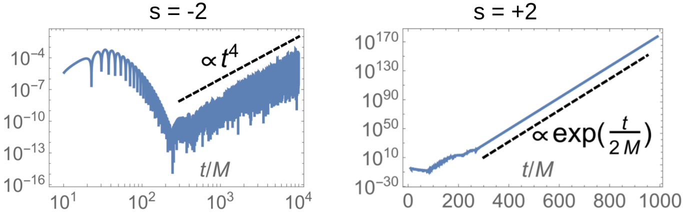
[Figure from arXiv:2406.08363]



- Cancellation problem necessitates rapid drop in $\ell_{\rm max}$ as r_p increases $\rightarrow$ TD code superior.
- Application of FD code to weak-field orbits $bv^2 \gg M$ less well-studied.
- TD/FD **hybridisation** sometimes useful [Long, Whittall & Barack 2406.08363].

Extension to gravity: TD code

- [Long & Barack 2105.05630] considered metric reconstruction for point mass moving along a scatter geodesic in Schwarzschild.
- Time-domain evolution for Hertz potentials $\phi_\pm$ obeying $s = \mp 2$ Teukolsky polluted by divergent unphysical modes:



- Remove using transformation to new variable $\phi \to X$ obeying Regge-Wheeler (RW) equation (Schwarzschild only)
- SF calculation needs five numerical derivatives of X:

$$\text{Solve RW for } X \xrightarrow{\partial^2} \text{Hertz potential } \phi \xrightarrow{\partial^2} h_{\alpha\beta} \xrightarrow{\partial} F_\mu .$$

Hyperboloidal slicing

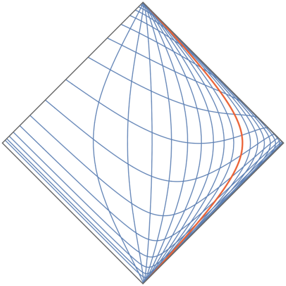
- Change to compactified hyperboloidal coordinates $(t, r) \mapsto (\tau, \sigma)$:

$$t = \tau + \frac{1}{\sigma} - 2\log[\sigma(1-\sigma)]; \quad r_* = r_*^p(t) + \frac{1}{\sigma} - 2 + 2\log[(1-\sigma)/\sigma]$$

- Fixed positions:

$$\mathcal{I}^+ : \sigma = 0 \qquad \mathcal{H}^+ : \sigma = 1 \qquad r_p : \sigma = \frac{1}{2}$$

- Compactification under-resolves wave-zone. Expected to eliminate unphysical incoming radiation.
- Two approaches (in time-domain) under development:
  - Spectral collocation scheme [Macedo, Long, Barack]
  - Finite differences [Vaswani & Barack]



[Image credit: O. Long]

Extension to gravity: FD code

- Techniques for scalar-field scattering in principle extend to gravitational calculations
  - Teukolsky solver + metric reconstruction
  - Solve for Lorenz gauge metric perturbation [Ackay, Warburton & Barack 2013]

- No external EHS: standard radiation gauge approach uses two-sided mode-sum regularisation. [Pound, Merlin & Barack 2014]
  - Lorenz gauge reconstruction? [Wardell, Kavanagh & Dolan 2024, ...]

- Cancellation problem: loss of precision at large radii will restrict accuracy of applications e.g. scatter angle.

Deficiencies of EHS complicate extension to gravity, limits potential

Developments in the frequency-domain

- "Nature adores a vacuum": use the form of the EHS on either side of the orbit as an ansatz, determine coefficients using junction conditions at the worldline [Dolan+ (under development)]
  - Reported to work well for low-eccentricity bound orbits.
  - Interest in scatter applications.
  - Form of $r > r_p(t)$ ansatz unclear (learning opportunity?)
  - Cancellation problem still occurs.
    - → Only useful if it proves much more accurate than evaluating radial integrals.

- Gibbs complementary reconstruction: reproject partial Fourier representation onto a "complementary" basis of polynomials.

Gibbs-complementary reprojection

- Say that a basis of functions $\{C_k\}$ (on a compact time interval I) is complementary to the Fourier basis if: [Gottlieb & Shu 1998]
  1. for any function f which is analytic on I, the expansion in the $\{C_k\}$ basis converges exponentially to f, **and**
  2. the projection of the high-frequency Fourier content onto the low-degree C_k can be made exponentially small.

- Gegenbauer polynomials $\left\{C_k^\lambda(s)\right\}$ orthogonal on $[-1,1]$ wrt weighted inner product,

$$\int_{-1}^{+1}(1-s^2)^{\lambda-1/2}C_n^\lambda(s)C_m^\lambda(s)ds = h_n^\lambda\delta_{nm}.$$

  - Generalisation of Legendre ($\lambda = 1/2$) and Chebyshev ($\lambda = 0, 1$) polynomials.
  - Complementary to the Fourier basis.

Gegenbauer reconstruction [Gottlieb & Shu 1992 (et al), 1994, 1995, 1997]

Suppose $\psi_{\ell m}(t,r)$ is analytic (at fixed r) on interval $a \leq t \leq b$:

1. Compute the partial Fourier integrals using the inhomogeneous modes $\psi_{\ell m\omega}(r)$ for $t \in [a,b]$:

$$\Psi_{\ell m}(t,r;\omega_{\mathrm{max}}) := \int_{-\omega_{\mathrm{max}}}^{+\omega_{\mathrm{max}}} \psi_{\ell m\omega}(r) e^{-i\omega t} d\omega.$$

2. Project onto the Gegenbauer basis:

$$g_k^\lambda(r;\omega_{\mathrm{max}}) := \frac{1}{h_k^\lambda} \int_{-1}^{1} (1-s^2)^{\lambda-1/2} \Psi_{\ell m}\left(t(s), r; \omega_{\mathrm{max}}\right) C_k^\lambda(s) ds.$$

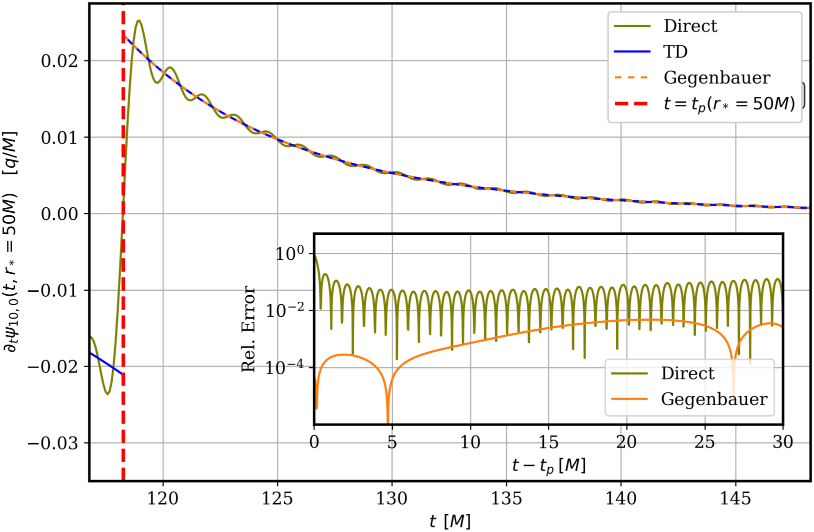
   where $t(s) = \left[(b-a)s + (a+b)\right]/2$.

3. Approximate

$$\psi_{\ell m}(t,r) \approx \sum_{k=0}^{N} g_k^\lambda C_k^\lambda(s(t)).$$

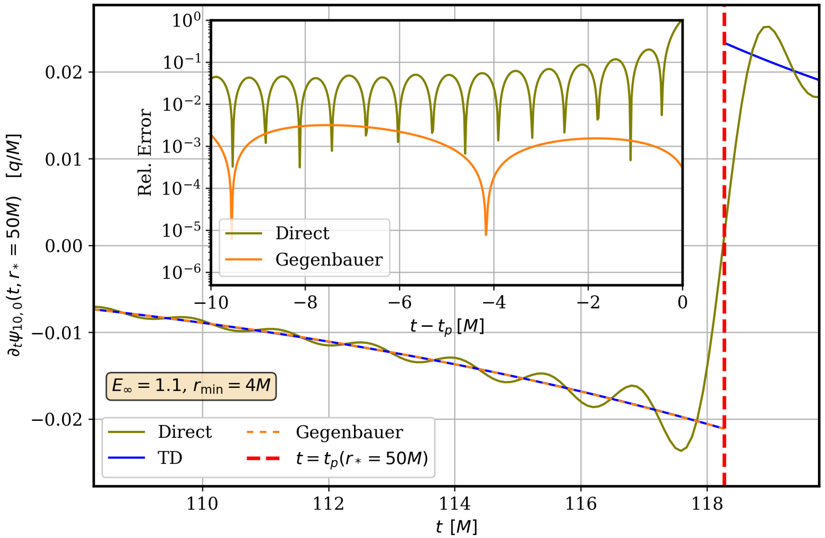
Gegenbauer approximant converges uniformly and exponentially on $a \leq t \leq b$, provided N, λ and $\omega_{\mathrm{max}} \to \infty$ in linear proportion.

Internal reconstruction: $r \leq r_p(t)$ [Whittall, Barack & Long in prep]



Gegenbauer reconstruction effective and outperforms Direct reconstruction.

External reconstruction: $r \geq r_p(t)$ [Whittall, Barack & Long in prep]



Gegenbauer reconstruction enables calculations in $r > r_p(t)$.

Gegenbauer reconstruction: discussion

- Circumvents Gibbs phenomenon and EHS challenges (cancellation, external reconstruction).
- Choice of N, λ impacts accuracy of approximant.
  - Robust parameter selection is key challenge for implementation.
- Computational cost: more integrals!
  - Gegenbauer reconstruction *will* be more expensive than EHS approach, but probably still feasible.
- Proof of concept scalar field calculations coming soon to arXiv.
- Next steps: EHS/Gegenbauer hybrid which switches from EHS to Gegenbauer at large radius.
- Other approaches exist! Large body of under-exploited literature on the Gibbs phenomenon.

Gravitational self-force fluxes

- Can compute gravitational fluxes in Schwarzschild from Regge-Wheeler-Zerilli equation, given by:

$$\frac{d^2 R_{\ell m\omega}}{dr_*^2} - \left(V_\ell^{\mathrm{RW/Z}}(r) - \omega^2\right) R_{\ell m\omega} = S_{\ell m\omega}^{\mathrm{RW/Z}}(r)$$

  in the frequency domain.

- Energy fluxes (for example) given by

$$\Delta E_{\ell m}^{\pm} = \frac{1}{64}\frac{(\ell+2)!}{(\ell-2)!}\int_{-\infty}^{+\infty} \omega^2 |C_{\ell m\omega}^{\pm}|^2 d\omega,$$

  where

$$C_{\ell m\omega}^{\pm} = \int_{r_{\min}}^{+\infty} \frac{R_{\ell\omega}^{\pm}(r') S_{\ell m\omega}^{\mathrm{RW/Z}}(r')}{W_{\ell\omega} f(r')} dr'.$$

Gravitational fluxes: scattering

- Previously computed for point mass scattering by [Hopper & Cardoso 2018, Hopper 2018].
- Recently repeated using same methodology by [Warburton (in prep)] (see next talk for results).
- Flux calculation can be seen as starting point for SF calculation: $C^{-}_{\ell m\omega}$ is the internal EHS normalization integral.
  - Flux calculation makes use of asymptotic fields: no EHS.
  - Self-force calculation needs field in vicinity of worldline: uses EHS.

Conclusions

Status

- Mature time- and frequency-domain codes for scalar-field self-force
- Extension to gravity not straightforward for either.
  - Unphysical Teukolsky modes (TD code)
  - Deficiencies of EHS (FD code)
- Various new techniques under development to resolve these issues.

Outlook

- Expect improved scalar-field scatter calculations soon.
- Gravitational flux calculations underway.
  - Might form basis for future (EHS-based) self-force calculation.
    - $\rightarrow$ Liable to the same problems facing the EHS scalar-field code.