Black hole scattering near the transition to plunge:

Self-force and resummation of post-Minkowskian theory

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Gravitational scattering

Why study scattering?

- Clean, well-defined asymptotic in/out states.
- Probe strong-field (sub-ISCO) at low energy.
- EOB Hamiltonian $H_{\rm EOB}$ completely determined by scatter angle $\chi(E, L)$. [Damour 2016+]
- B2B maps between scatter and bound orbit. [Kalin & Porto 2020+]

Complementary approaches:

- **Post-Minkowskian expansion (expansion in G):** weak-field, arbitrary mass-ratio, analytical.
- **Gravitational self-force (expansion in ϵ):** strong-field, small mass-ratio, numerical.

GSF can be used to calibrate/benchmark PM in the strong-field

Gravitational self-force

Metric of the physical spacetime is expanded about background as a series in $\epsilon := \mu/M \ll 1$,

$$g^{\mathrm{phys}}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots$$

- **0SF**: Background metric $g_{\alpha\beta}$. Smaller object moves along fixed background geodesic.
- **1SF**: Perturbation $h^{(1)}_{\alpha\beta}$ sourced by point particle on fixed background geodesic. Leading order self-force $\propto \epsilon$:

$$\frac{Du^\alpha}{d\tau} = -\frac{\epsilon}{2} g^{\alpha\beta}\left(2h^{(1)\mathrm{R}}_{\beta\rho;\sigma} - h^{(1)\mathrm{R}}_{\rho\sigma;\beta}\right) u^\rho u^\sigma + O(\epsilon^2) := F^\alpha/\mu, \qquad (1)$$

Scalar-field self-force and scattering

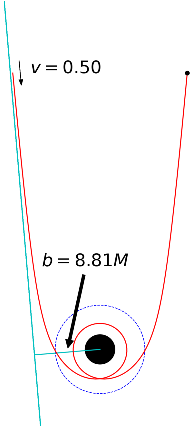
- Geodesic scatter orbits parameterised by velocity at infinity v and impact parameter $b > b_c(v)$.

- Using scalar-field toy model in Schwarzschild

$$\nabla\Phi = -4\pi q \int \frac{\delta^4(x^\alpha - x_p^\alpha(\tau))}{\sqrt{-g}} d\tau,$$
$$u^\beta \nabla_\beta u^\alpha = q(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta \Phi^R := \epsilon F^\alpha,$$

where $\epsilon := q^2/\mu M \ll 1$ is the expansion parameter.

- Self-force can be split into conservative (time-symmetric) and dissipative (antisymmetric) pieces: $F^\alpha = F^\alpha_{\mathrm{cons}} + F^\alpha_{\mathrm{diss}}$.



SF scatter angle correction

- Scatter angle defined

$$\chi := \varphi_{\rm out} - \varphi_{\rm in} - \pi.$$

- Self-force expansion:

$$\chi(v, b) = \chi^{\rm 0SF}(v, b) + \epsilon\chi^{\rm 1SF}(v, b) + O(\epsilon^2),$$

N.B.: split between geodesic term $\chi^{\rm 0SF}$ and self-force *correction* $\chi^{\rm 1SF}$ defined at fixed (v, b).

- 1SF correction expressed as integral of SF along background geodesic [Barack & Long 22]

$$\chi^{\rm 1SF} = \int_{-\infty}^{+\infty} \left[\mathcal{G}_E(\tau)F_t(\tau) - \mathcal{G}_L(\tau)F_\varphi(\tau)\right] d\tau$$

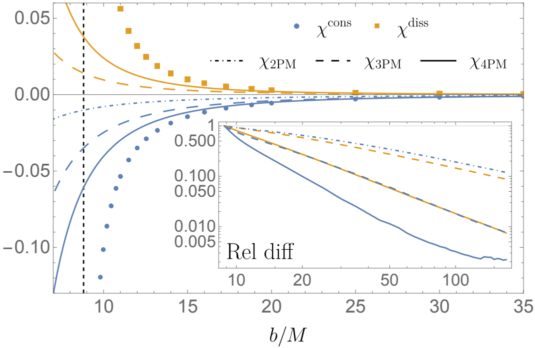
PM expansion of $\chi^{1\mathrm{SF}}$

- Analytical progress using post-Minkowskian expansion,

$$\chi^{1\mathrm{SF}} = \sum_{k=2}^{\infty} \chi_k^{1\mathrm{SF}}(v) \left(\frac{GM}{b}\right)^k .$$

- Coefficients known through 4PM order for scalar-field. [Gralla & Lobo 22, Barack & Long 22, Barack et al 23, Bini et al 24]

- Good agreement found with numerical self-force in weak-field $b \to \infty$. [Barack et al 23]



Transition to plunge

- Separatrix $b = b_c(v)$ divides scatter ($b > b_c(v)$) from plunge ($b < b_c(v)$).
- Each critical "geodesic" $b = b_c(v)$ has two branches:
  - **Inbound:** begins at infinity, is captured into circular orbit.
  - **Outbound:** begins as circular orbit, escapes to infinity.
- Conservative/dissipative forces obtained from combinations of SF along inbound/outbound branches.

$$\delta b := b - b_c(v)$$

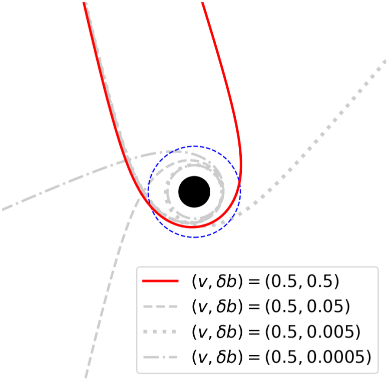


Figure: scatter geodesics in the $b \to b_c(v)$ limit.

Singularity structure of $\chi^{0\mathrm{SF}}$ and $\chi^{1\mathrm{SF}}$

- Log divergence in $\chi^{0\mathrm{SF}}$:

$$\chi^{0SF} \sim A_0(v) \log\left(\frac{\delta b}{b_c(v)}\right) \text{ as } b \to b_c(v),$$

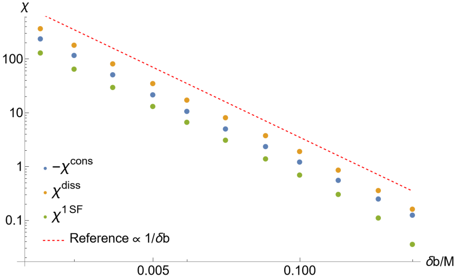
where, recall, $\delta b := b - b_c(v)$, and

$$A_0(v) = -\left(1 - \frac{12M^2(1-v^2)}{v^2 b_c(v)^2}\right)^{1/2}.$$

- Faster divergence at 1SF,

$$\chi^{1SF} \sim A_1(v) \frac{b_c(v)}{\delta b},$$

as $b \to b_c(v)$. [Barack & Long 2022]



Integral expression for $A_1(v)$ along critical orbit

Divergence parameters $A_1^{\mathrm{cons/diss}}(v)$ can be expressed

$$A_1^{\mathrm{cons}}(v) = -\frac{1}{b_c(v)} \int_{-\infty}^{+\infty} \left(c_E F_t^{\mathrm{cons}} + c_L F_\varphi^{\mathrm{cons}}\right) d\tau,$$
$$A_1^{\mathrm{diss}}(v) = \frac{1}{b_c(v)} \int_{-\infty}^{+\infty} \left(c_E F_t^{\mathrm{diss}} + c_L F_\varphi^{\mathrm{diss}}\right) d\tau,$$

where the integrals and self-forces are evaluated on the *outbound* critical orbit and $c_{E/L}$ are constants.

- Calculation confirms $1/\delta b$ divergence analytically.
- For each v, $A_1^{\mathrm{cons}}(v)$ and $A_1^{\mathrm{diss}}(v)$ obtained in principle by SF calculation along only 2 orbits.

SF-informed PM resummation

- Introduce

$$\Delta\chi(v,b) := A_0\left[\log\left(1-\frac{b_c(v)(1-\epsilon A_1/A_0)}{b}\right)+\sum_{k=1}^{4}\frac{1}{k}\left(\frac{b_c(v)(1-\epsilon A_1/A_0)}{b}\right)^k\right].$$

  - $\Delta\chi = O(b^{-5})$ as $b\to\infty$
  - Matches the $b\to b_c(v)$ divergences of $\chi(v,b)$ at both 0SF and 1SF.

- Resummed scatter angle:

$$\tilde{\chi}(v,b) := \chi_{4\mathrm{PM}}(v,b)+\Delta\chi(v,b).$$

  - Matches $b\to\infty$ behaviour of χ through 4PM order.
  - Matches $b\to b_c(v)$ behaviour at 0SF and 1SF.

- Similar to geodesic order approach introduced in [Damour & Rettegno 2023], but extended to 1SF.

Numerical platforms

Two codes available for scalar-field self-force scatter calculations:

- **Time-domain** code: [Barack & Long 2209.03740+]
  - Finite differences, null grid.
  - Performed first calculations of χ^{1SF}.
  - Typically limited to $\ell_{\max} = 15$.
- **Frequency-domain** code: [Whittall & Barack 2305.09724]
  - SF reconstructed from frequency modes.
  - Highly accurate near periapsis, access to at least $\ell_{\max} = 25$.
  - Loss of precision for large-ℓ modes at larger radii; $\ell_{\max}$ must be reduced rapidly.

FD code more accurate than TD near periapsis, less accurate further away.

Both codes restricted to non-critical orbits - must extrapolate from $\delta b > 0$!

High-velocity limit

- Large-ℓ modes become more important at high velocities.
- Delayed transition to asymptotic behaviour in mode-sum.
  - Possibly associated with relativistic beaming.
- $\ell > 15$ modes can contribute up to a few percent of the total SF.

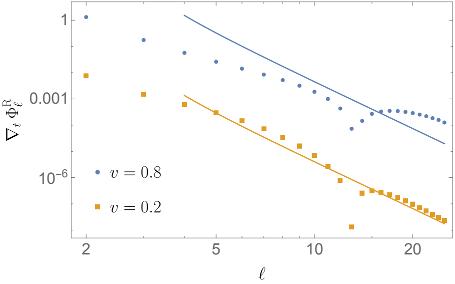


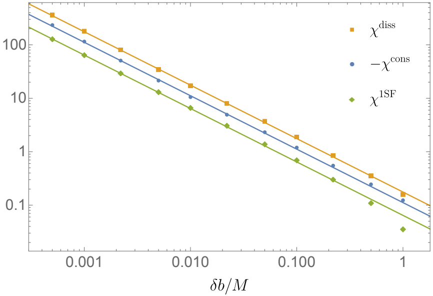
Figure: regularised ℓ-mode contributions to $\nabla_t \Phi^R$ at given points along example low and high velocity orbits.

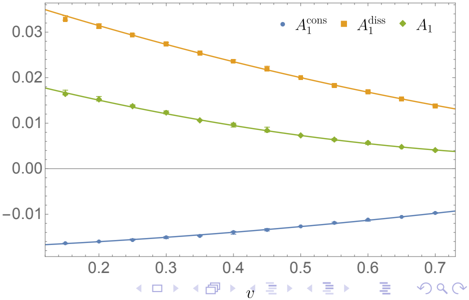
- Effect largest near periapsis, where FD code can handle $\ell_{\max} > 15$.
- Motivated development of TD/FD hybrid approach.

Calculating $A_1(v)$ by extrapolation

- Calculate $A_1(v)$ by extrapolating along sequence of ~ 10 orbits with $b \to b_c(v)$ at fixed v.
- More accurate to fit for $A_1^{\mathrm{cons/diss}}(v)$ separately.
- Fits performed in Mathematica, weighting each scatter angle by $1/\epsilon^2_{\mathrm{num}}$.
- Effect of varying number of points included in fit: investigated and incorporated into error bars on $A_1^{\mathrm{cons/diss}}$.

Figure: fit models $A_1(v)b_c(v)/\delta b$ for $v = 0.5$.





Resummation: 1SF scatter angle correction [OL, CW & LB 2406.08363]

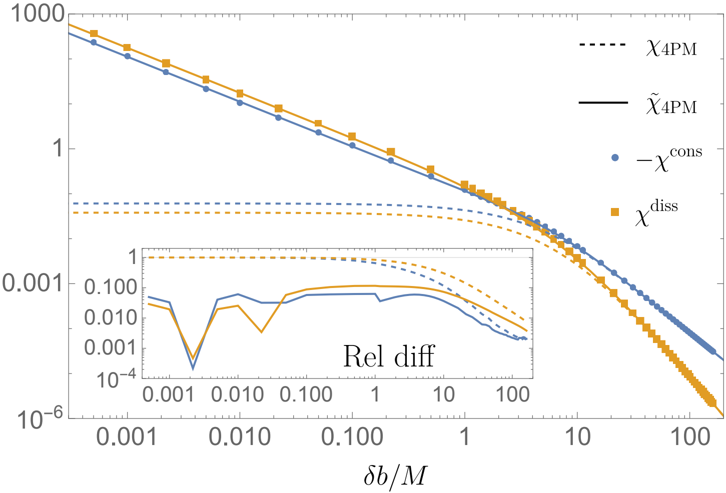


Figure: the resummation procedure significantly improves agreement with the numerical SF data, even in the weak-field. ($v = 0.5$)

Resummation: total scatter angle [OL, **CW** & LB 2406.08363]

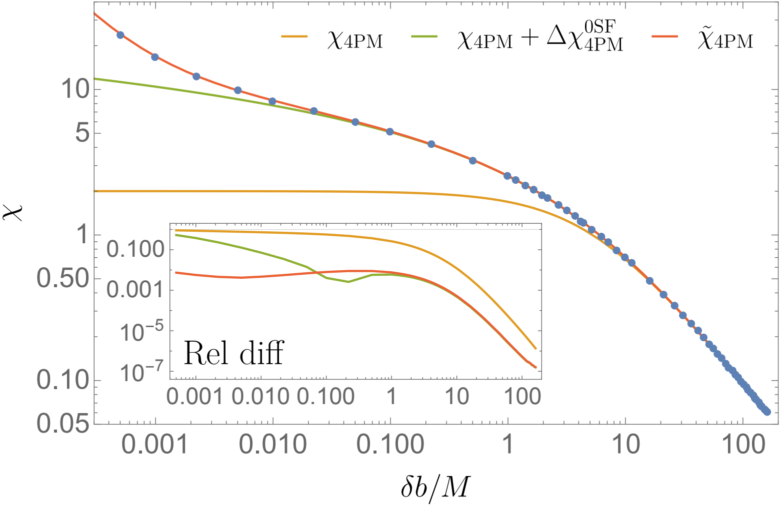


Figure: $\chi^{0\mathrm{SF}} + 0.1\chi^{1\mathrm{SF}}$ for $v = 0.5$. Our 1SF resummation improves upon the geodesic order resummation in the $\delta b \to 0$ limit.

Summary and outlook

Resummed PM provides semi-analytical model which is fast to evaluate and accurate in both strong and weak-field at 1SF.

Next steps:

- Direct calculation of $A_1(v)$ as integral over critical orbit should increase accuracy and decrease computational burden.
  - Interesting distributional frequency spectrum for critical orbit.
- Framework easily extends to gravity once GSF available.