Frequency-domain self-force calculations using Gegenbauer reconstruction

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CAPRA

Gravitational scattering

Why study scattering?

- Clean, well-defined asymptotic in/out states.
- Probe strong-field (sub-ISCO) at low energy.
- EOB Hamiltonian H_{EOB} completely determined by scatter angle $\chi(E, L)$. [Damour 2016+]
- B2B maps between scatter and bound orbit. [Kalin & Porto 2020+]

Complementary approaches include:

- **Numerical relativity:** low-separation, short durations, $\approx$ equal mass.
- **Post-Minkowskian expansion (expansion in G):** weak-field, arbitrary mass-ratio, analytical.
- **Gravitational self-force (expansion in mass-ratio ϵ):** strong- and weak-field, small mass-ratio, numerical.

Interest in developing numerical GSF calculations along scatter orbits.

Scalar-field self-force and scattering

- Scalar charge q with mass μ scattered off a Schwarzschild black hole of mass M. Scalar field:

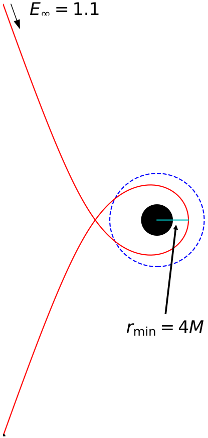
$$\nabla\Phi = -4\pi q \int \frac{\delta^4(x^\alpha - x_p^\alpha(\tau))}{\sqrt{-g}} d\tau,$$

where $\epsilon := q^2/\mu M \ll 1$ is the expansion parameter.

- At leading order take $x_p^\alpha(\tau)$ to be a scatter geodesic: parameterised by energy at infinity, E_∞, and periapsis radius $r_{\min}$.

- Particle feels a self-force due to interaction with its own scalar field:

$$u^\beta \nabla_\beta (\mu u^\alpha) = q \nabla_\beta \Phi^R := F^\alpha.$$



Numerical self-force calculations

Decompose the field into spherical harmonics centred on the Schwarzschild black hole:

$$\Phi(t,r,\theta,\phi) = \frac{1}{r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\psi_{\ell m}(t,r)Y_{\ell m}(\theta,\phi)$$

Self-force can be computed using mode-sum regularisation:

$$F_\alpha(\tau) = \sum_{\ell=0}^{\infty}\left\{\nabla_\alpha\left[\frac{1}{r}\sum_{m=-\ell}^{+\ell}\psi_{\ell m}Y_{\ell m}\right]_{x_p^{\pm}(\tau)} \mp \left(\ell+\frac{1}{2}\right)\underbrace{A_\alpha(\tau)-B_\alpha(\tau)}_{\text{regularisation parameters}}\right\}$$

Time-domain

- Solve $(1+1)d$ PDEs for $\psi_{\ell m}(t,r)$.
- Scattering: [Long & Barack 2209.03740]

Frequency-domain

- Construct $\psi_{\ell m}(t,r)$ from frequency-modes, obtained by solving ODEs.
- Scattering: [Whittall & Barack 2305.09724]

Direct reconstruction

- Frequency-modes $\psi_{\ell m\omega}$ obey ODE

$$\frac{d^2\psi_{\ell m\omega}}{dr_*^2} - \left[V_\ell(r) - \omega^2\right]\psi_{\ell m\omega} = S_{\ell m\omega}(r)$$

- Retarded solution expressed in terms of homogeneous solution basis $\psi^{\pm}_{\ell\omega}(r)$ using variation of parameters:

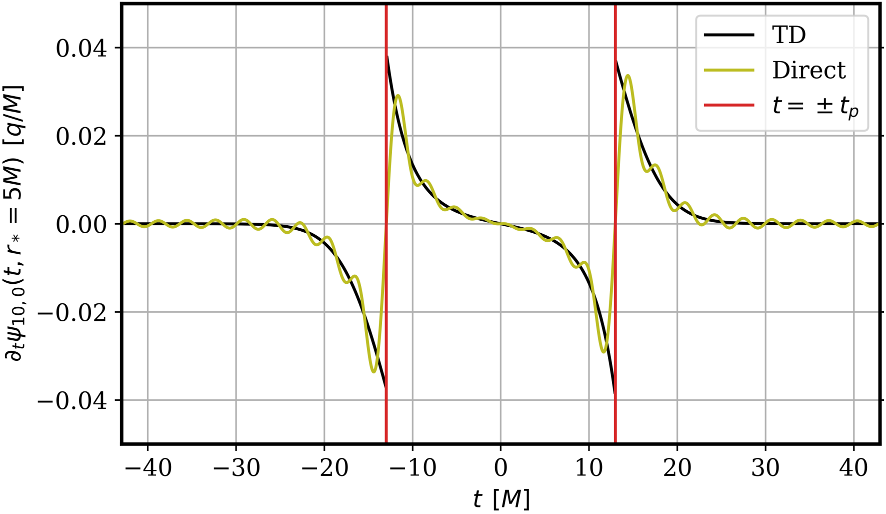
$$\psi_{\ell m\omega}(r) = \psi^{+}_{\ell\omega}(r)\int_{r_{\min}}^{r} \frac{\psi^{-}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega}f(r')}dr' + \psi^{-}_{\ell\omega}(r)\int_{r}^{+\infty} \frac{\psi^{+}_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega}f(r')}dr'$$

- Direct reconstruction:

$$\psi_{\ell m}(t,r) \approx \Psi_{\ell m}(t,r;\omega_{\max}) := \int_{-\omega_{\max}}^{+\omega_{\max}} \psi_{\ell m\omega}(r)e^{-i\omega t}d\omega$$

is **not practical** due to the Gibbs phenomenon.

Direct reconstruction



EHS reconstruction [Barack, Ori & Sago 2008]

- EHS reconstruction: recover $\psi_{\ell m}(t,r)$ separately in $r \leq r_p(t)$ and $r \geq r_p(t)$ using homogeneous solutions.
- For example, field modes in the "internal" region $r \leq r_p(t)$ reconstructed from

$$\tilde{\psi}^-_{\ell m\omega}(r) := \psi^-_{\ell\omega}(r) \int_{r_{\min}}^{+\infty} \frac{\psi^+_{\ell\omega}(r')S_{\ell m\omega}(r')}{W_{\ell\omega}f(r')} dr'.$$

- Restores exponential, uniform convergence.
- External reconstruction: EHS cannot be applied in $r > r_p(t)$ for scatter orbits.
- Cancellation problem: significant loss of precision at high eccentricities [van de Meent 2016]. Limited to $r_p \sim r_{\min}$ for scatter orbits.

Gibbs-complementary reprojection

- Say that a basis of functions $\{C_k\}$ (on a compact time interval I) form a Gibbs-complement to the Fourier basis if: [Gottlieb & Shu 1998]
  1. for any function f which is analytic on I, the expansion in the $\{C_k\}$ basis converges exponentially to f, **and**
  2. the projection of the high-frequency Fourier content onto the low-degree C_k can be made exponentially small.

- Gegenbauer polynomials $\left\{C_k^\lambda(s)\right\}$ orthogonal on $[-1, 1]$ wrt weighted inner product,

$$\int_{-1}^{+1} (1 - s^2)^{\lambda - 1/2} C_n^\lambda(s) C_m^\lambda(s) ds = h_n^\lambda \delta_{nm}.$$

  - Generalisation of Legendre ($\lambda = 1/2$) and Chebyshev ($\lambda = 0, 1$) polynomials.
  - Gibbs-complementary to the Fourier basis.

Gegenbauer reconstruction [Gottlieb & Shu 1992 (et al), 1994, 1995, 1997]

Suppose $\psi_{\ell m}(t,r)$ is analytic (at fixed r) on interval $a \leq t \leq b$:

1. Compute the partial Fourier integrals using the inhomogeneous modes $\psi_{\ell m \omega}(r)$ for $t \in [a,b]$:

$$\Psi_{\ell m}(t,r;\omega_{\max}) := \int_{-\omega_{\max}}^{+\omega_{\max}} \psi_{\ell m \omega}(r) e^{-i\omega t} d\omega.$$

2. Project onto the Gegenbauer basis:

$$g_k^{\lambda}(r;\omega_{\max}) := \frac{1}{h_k^{\lambda}} \int_{-1}^{1} (1-s^2)^{\lambda-1/2} \Psi_{\ell m}\left(t(s), r; \omega_{\max}\right) C_k^{\lambda}(s) ds.$$

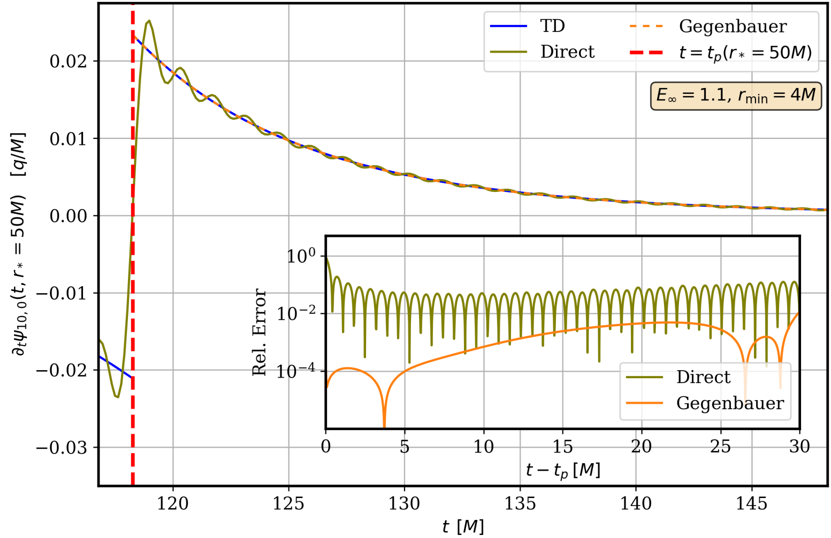
where $t(s) = \left[(b-a)s + (a+b)\right]/2$.

3. Approximate

$$\psi_{\ell m}(t,r) \approx \sum_{k=0}^{N} g_k^{\lambda} C_k^{\lambda}(s(t)).$$

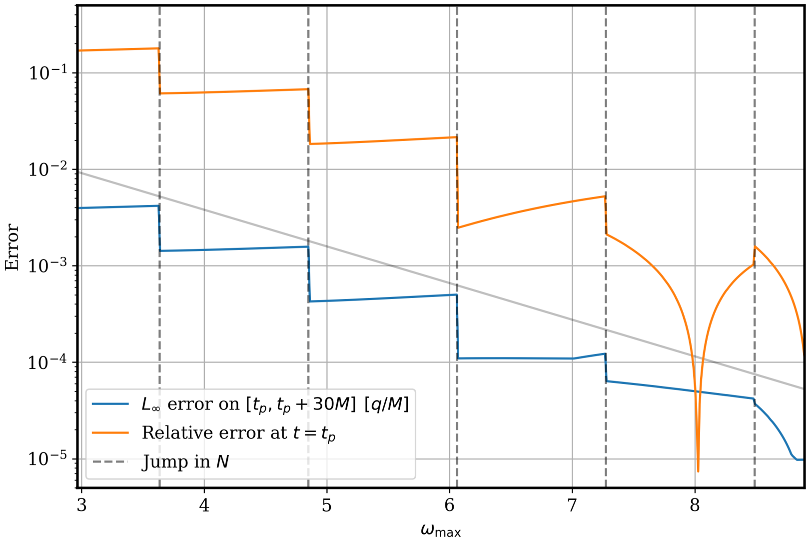
Gegenbauer approximant converges uniformly and exponentially on $a \leq t \leq b$, provided N, λ and $\omega_{\max} \to \infty$ in linear proportion.

Internal reconstruction: $r \leq r_p(t)$



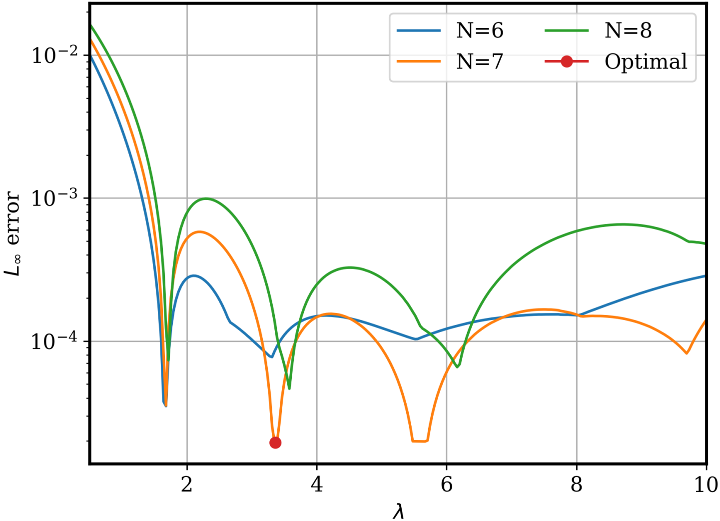
Gegenbauer reconstruction effective and outperforms Direct reconstruction.

Convergence



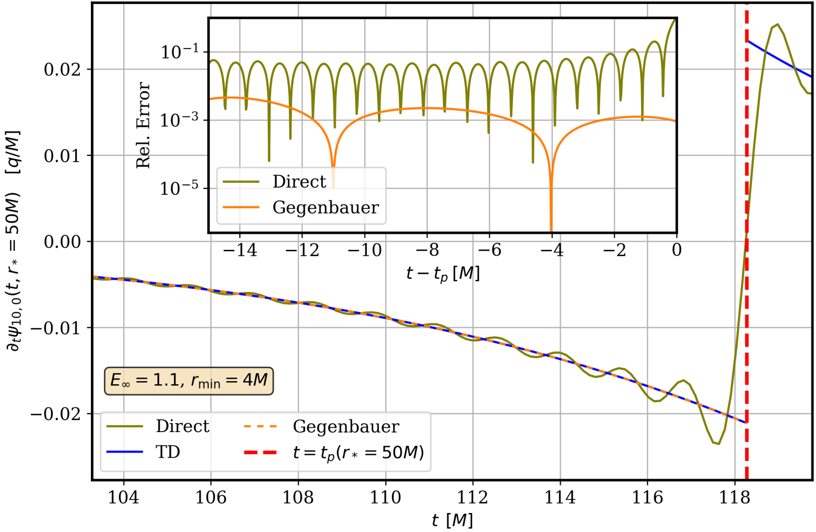
Convergence expected to be uniform and exponential.

Choice of N and λ



Parameters N and λ must be chosen appropriately.

External reconstruction: $r \geq r_p(t)$



Gegenbauer reconstruction enables calculations in $r > r_p(t)$.

Concluding remarks

- **Coming soon to arXiv:** proof of concept calculations for scalar-field modes $\psi_{\ell m}(t, r)$ with scattering source.
- Working towards full scalar-field self-force scatter calculation:
  - Computational efficiency: reconstruction, radial integrals in FD.
    - $\rightarrow$ TD reconstruction cheaper than FD mode calculation.
    - $\rightarrow$ Economise FD calculation by reusing integrals/integrands, or use ODE integration.
  - Optimal choices of $N/\omega_{\max}$ and $\lambda/\omega_{\max}$.
    - $\rightarrow$ Calibrate to EHS then update using convergence tests?
    - $\rightarrow$ Choices unlikely to be optimal, but may be adequate.
- Gravity is the ultimate aim .
- Advanced reprojection methods?
  - Alternative bases (e.g. Freud polynomials [Gelb & Tanner 2006]).
  - Inverse polynomial reconstruction [Shizgal & Jung 2003]
  - ... and more! e.g. [Adcock & Hansen 2011]
- New applications?
  - Easy to implement as post-processing step.